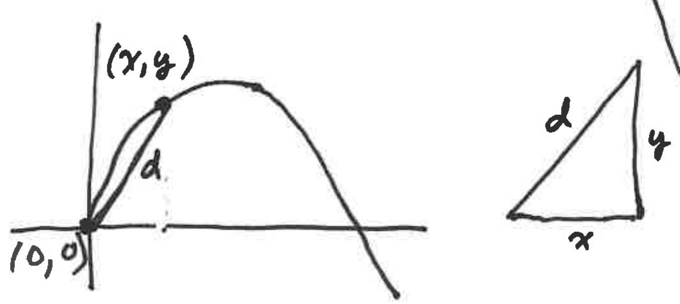
1a) Give an expression in terms of x that finds the rate of change of the distance between the origin and a moving point on the graph of $y = x^2 + 1$ if $\frac{dx}{dt} = 2\text{cm}/\text{sec}$

$$D = \sqrt{x^2 + y^2} = \sqrt{x^2 + (x^2+1)^2} = \sqrt{x^4 + 3x^2 + 1}$$

$$\frac{dx}{dt} = 2$$

$$\frac{dD}{dt} = \frac{1}{2}(x^4 + 3x^2 + 1)^{-1/2}(4x^3 + 6x)\frac{dx}{dt} = \frac{2x^3 + 3x}{\sqrt{x^4 + 3x^2 + 1}}\frac{dx}{dt} = \frac{4x^3 + 6x}{\sqrt{x^4 + 3x^2 + 1}}$$

2b) Give an expression in terms of x that finds the rate of change of the distance $\frac{dd}{dt}$ between the origin and a moving point on the graph of $y = \sin x$ if $\frac{dx}{dt} = 2\text{cm}/\text{sec}$



$$x^2 + y^2 = d^2$$

$$x^2 + (\sin x)^2 = d^2$$

$$2x\frac{dx}{dt} + 2(\sin x)\cdot\cos x\frac{dx}{dt} = 2d\frac{dd}{dt}$$

$$\frac{dx}{dt}(2x + \sin 2x) = 2d\frac{dd}{dt}$$

by identity

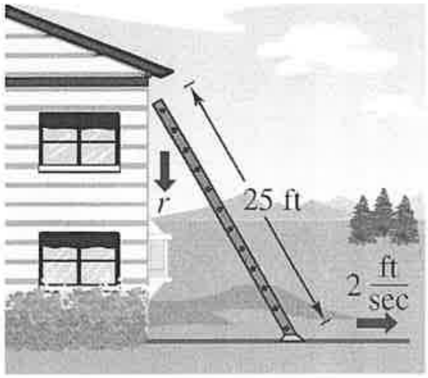
$$\frac{\frac{dx}{dt}(2x + \sin 2x)}{2d} = \frac{dd}{dt}$$

$$\frac{\cancel{2}(2x + \sin 2x)}{\cancel{2}d} = \frac{dd}{dt}$$

$$\boxed{\frac{2x + \sin 2x}{\sqrt{x^2 + \sin^2 x}} = \frac{dd}{dt}}$$

IF
$$x^2 + \sin^2 x = d^2$$
$$\sqrt{x^2 + \sin^2 x} = d$$

2a-c) A ladder 25 feet long is leaning against the wall of a house (see figure). The base of the ladder is pulled away from the wall at a rate of 2 feet per second.



(a) How fast is the top of the ladder moving down the wall when its base is 7 feet from the wall?

$$x^2 + y^2 = 25^2$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = \frac{-x}{y}\cdot\frac{dx}{dt} = \frac{-2x}{y} \text{ since } \frac{dx}{dt} = 2.$$

When $x = 7$, $y = \sqrt{576} = 24$, $\dfrac{dy}{dt} = \dfrac{-2(7)}{24} = \dfrac{-7}{12}$ ft/sec.

(b) Consider the triangle formed by the side of the house, the ladder, and the ground. Find the rate at which the area of the triangle is changing when the base of the ladder is 7 feet from the wall.

$$A = \frac{1}{2}xy$$

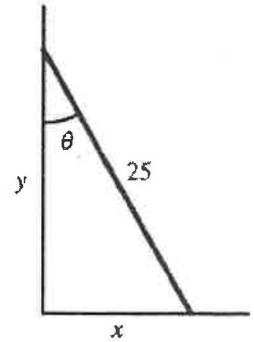
$$\frac{dA}{dt} = \frac{1}{2}\left(x\frac{dy}{dt} + y\frac{dx}{dt}\right)$$

From part (a) we have $x = 7$, $y = 24$, $\dfrac{dx}{dt} = 2$, and $\dfrac{dy}{dt} = -\dfrac{7}{12}$.

Thus, $\dfrac{dA}{dt} = \dfrac{1}{2}\left[7\left(-\dfrac{7}{12}\right) + 24(2)\right] = \dfrac{527}{24} \approx 21.96$ ft²/sec.

(c) Find the rate at which the angle between the ladder and the wall of the house is changing when the base of the ladder is 7 feet from the wall.

$$\tan\theta = \frac{x}{y}$$



$$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{y}\cdot\frac{dx}{dt} - \frac{x}{y^2}\cdot\frac{dy}{dt}$$

$$\frac{d\theta}{dt} = \cos^2\theta\left[\frac{1}{y}\cdot\frac{dx}{dt} - \frac{x}{y^2}\cdot\frac{dy}{dt}\right]$$

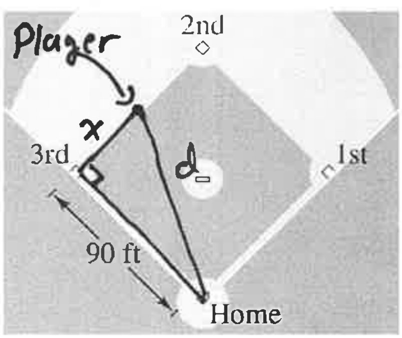
Using $x = 7, y = 24, \frac{dx}{dt} = 2, \frac{dy}{dt} = -\frac{7}{12}$ and $\cos\theta = \frac{24}{25}$, we have

$$\frac{d\theta}{dt} = \left(\frac{24}{25}\right)^2\left[\frac{1}{24}(2) - \frac{7}{(24)^2}\left(-\frac{7}{12}\right)\right] = \frac{1}{12}\text{ rad/sec.}$$

2d) A baseball diamond has the shape of a square with sides 90 feet long (see figure). A player running from second base to third base at a speed of 28 feet per second is 30 feet from third base. At what rate is the player's distance from home plate changing?

$\frac{dx}{dt} = -28$

x is distance from 3rd base



d is distance from home

$$x^2 + 90^2 = d^2$$

$$2x\frac{dx}{dt} = 2d\frac{dd}{dt}$$

$$\frac{\cancel{2}x\frac{dx}{dt}}{\cancel{2}d} = \frac{dd}{dt}$$

when $x = 30$

$30^2 + 90^2 = d^2$

$9000 = d^2$

$\sqrt{9000} = d$

$$\frac{30\cdot(-28)}{\sqrt{9000}} = \frac{dd}{dt}$$

$$\frac{30\cdot(-28)}{30\sqrt{10}} = \frac{dd}{dt}$$

$$\boxed{\frac{-28}{\sqrt{10}}\frac{\text{ft}}{\text{sec}} = \frac{dd}{dt}}$$

2c) A balloon rises at a rate of 3 meters per second from a point on the ground 30 meters from an observer. Find the rate of change of the angle of elevation of the balloon from the observer when the balloon is 30 meters above the ground.

$$\tan\theta = \frac{h}{30}$$

$$\sec^2\theta\frac{d\theta}{dt} = \frac{1}{30}\frac{dh}{dt}$$

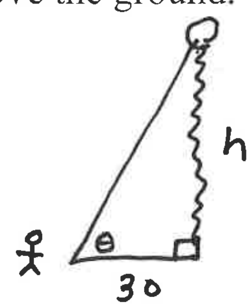
$$\frac{d\theta}{dt} = \frac{1}{30\sec^2\theta}\frac{dh}{dt}$$

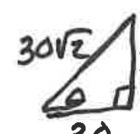
$$\frac{d\theta}{dt} = \frac{\cos^2\theta}{30}\frac{dh}{dt}$$

$$\frac{d\theta}{dt} = \frac{\left(\frac{1}{\sqrt{2}}\right)^2}{\cancel{30}_{10}}\cdot\cancel{3}^{1}$$

$$\frac{d\theta}{dt} = \frac{\frac{1}{2}}{10}$$

$$\boxed{\frac{d\theta}{dt} = \frac{1}{20}\text{ rad/sec}}$$





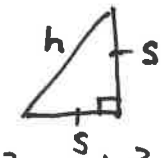
when $h = 30$

$\cos\theta = \frac{30}{30\sqrt{2}}$

3) AP MULTIPLE CHOICE EXAMPLES

1) An isosceles right triangle with legs of length s has area $A = \frac{1}{2}s^2$. At the instant when $s = \sqrt{32}$ centimeters, the area of the triangle is increasing at a rate of 12 square centimeters per second. At what rate is the length of the hypotenuse of the triangle increasing, in centimeters per second, at that instant?

$\frac{dA}{dt} = +12$



(A) $\frac{3}{4}$ (B) 3 (C) $\sqrt{32}$ (D) 48

$A = \frac{1}{2}s^2$

so.. $A = \frac{1}{2}\left(\frac{1}{2}h^2\right)$

$A = \frac{1}{4}h^2$

$\frac{dA}{dt} = \frac{1}{2}h\frac{dh}{dt}$

$\frac{2\frac{dA}{dt}}{h} = \frac{dh}{dt}$

$\frac{2 \cdot 12}{8} = \frac{dh}{dt}$

$3 = \frac{dh}{dt}$

$s^2 + s^2 = h^2$

$2s^2 = h^2$

$s^2 = \frac{h^2}{2}$

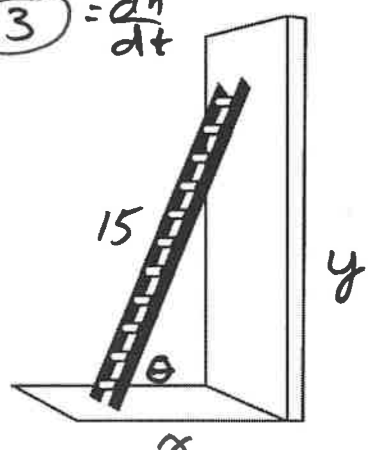
when $s = \sqrt{32}$: $2(\sqrt{32})^2 = h^2$

$2 \cdot 32 = h^2$

$64 = h^2$

$8 = h$

2)



The top of a 15-foot-long ladder rests against a vertical wall with the bottom of the ladder on level ground, as shown above. The ladder is sliding down the wall at a constant rate of 2 feet per second. At what rate, in radians per second, is the acute angle between the bottom of the ladder and the ground changing at the instant the bottom of the ladder is 9 feet from the base of the wall?

distance decreasing

$\frac{dy}{dt} = -2$

(A) $-\frac{2}{9}$ (B) $-\frac{1}{6}$ (C) $-\frac{2}{25}$ (D) $\frac{2}{25}$ (E) $\frac{1}{9}$

NOTE: WE DO NOT have $\frac{dx}{dt}$!

Using tangent is a bad idea!!

when $x = 9$
$\cos\theta = \frac{9}{15}$ or $\frac{3}{5}$

$\sin\theta = \frac{y}{15}$

$\cos\theta \frac{d\theta}{dt} = \frac{1}{15}\frac{dy}{dt}$

$\frac{d\theta}{dt} = \frac{1}{15\cos\theta}\frac{dy}{dt}$

$\frac{d\theta}{dt} = \frac{1}{15 \cdot \left(\frac{3}{5}\right)} \cdot (-2)$

$\frac{d\theta}{dt} = \frac{1}{9} \cdot -2$

$\boxed{\frac{d\theta}{dt} = -2/9 \text{ rad/sec}}$

3) Functions w, x, and y are differentiable with respect to time and are related by the equation $w = x^2 y$. If x is decreasing at a constant rate of 1 unit per minute and y is increasing at a constant rate of 4 units per minute, at what rate is w changing with respect to time when $x = 6$ and $y = 20$?

$\frac{dx}{dt} = -1$

$\frac{dy}{dt} = 4$

(A) -384 (B) -240 (C) -96 (D) 276 (E) 384

$$w = x^2 y$$

$$\frac{dw}{dt} = x^2 \frac{dy}{dt} + y \cdot 2x \frac{dx}{dt}$$

$$\frac{dw}{dt} = (6)^2 \cdot (4) + 20 \cdot 2 \cdot (6) \cdot (-1)$$

$$\therefore = 144 + (-240)$$

$$\therefore = -96$$

4) The radius of a circle is increasing at a nonzero rate, and at a certain instant, the rate of increase in the area of the circle is numerically $\boxed{\text{equal}}$ to the rate of increase in its circumference. At this instant, the radius of the circle is

(A) $\frac{1}{\pi}$ (B) $\frac{1}{2}$ (C) $\frac{2}{\pi}$ $\boxed{\text{(D) } 1}$ (E) 2

$A = \pi r^2$ $\quad C = 2\pi r$

$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$ $\quad \frac{dC}{dt} = 2\pi \frac{dr}{dt}$

if $\frac{dA}{dt} = \frac{dC}{dt}$

then

$$2\pi r \frac{dr}{dt} = 2\pi \frac{dr}{dt}$$

$$r = \frac{\cancel{2\pi \frac{dr}{dt}}}{\cancel{2\pi \frac{dr}{dt}}}$$

$$r = 1$$