AP Calculus Inverse Trig. Functions and Integration Critical Homework

Integrate each function.

1a) $\int \frac{dx}{\sqrt{9-x^2}}$

$$\int \frac{dx}{\sqrt{9-x^2}} = \arcsin\left(\frac{x}{3}\right) + C$$

1b) $\int \frac{7}{16+x^2}\,dx$

$$= 7\int \frac{1}{16+x^2}\,dx$$

$a=4$

$$= 7\int \frac{1}{a^2+x^2}\,dx$$

$$= 7 \cdot \frac{1}{4}\text{Arctan}\left(\frac{x}{4}\right)+c$$

$$= \boxed{\frac{7}{4}\text{Arctan}\left(\frac{x}{4}\right)+c}$$

1c) $\int \frac{1}{x\sqrt{4x^2-1}}\,dx$

$a=1$

$u=2x$

$\frac{du}{dx}=2$

$\frac{1}{2}du=dx$

$$\int \frac{1}{x\sqrt{u^2-a^2}} \cdot \frac{1}{2}du$$

$$\int \frac{1}{2x\sqrt{u^2-a^2}}\,du$$

$$\int \frac{1}{u\sqrt{u^2-a^2}}\,du$$

$$\frac{1}{1}\text{Arcsec}\,|2x|+C$$

$$\boxed{\text{Arcsec}\,|2x|+C}$$

2a) $\int \frac{t}{\sqrt{1-t^4}}\,dt$

Let $u = t^2$, $du = 2t\,dt$

$$= \frac{1}{2}\int \frac{1}{\sqrt{1-(t^2)^2}}(2t)\,dt$$

$$= \frac{1}{2}\arcsin t^2 + C$$

$a=5$

$u=t^2$

$\frac{du}{dt}=2t$

$\frac{1}{2t}du=dt$

2b) $\int \frac{t}{t^4+25}\,dt$

$$\int \frac{\cancel{t}}{u^2+a^2}\cdot\frac{1}{2\cancel{t}}\,du$$

$$\frac{1}{2}\int \frac{1}{u^2+a^2}\,du$$

$$\frac{1}{2}\cdot\frac{1}{a}\text{Arctan}\left(\frac{u}{a}\right)+c$$

$$\boxed{\frac{1}{10}\text{Arctan}\left(\frac{t^2}{5}\right)+c}$$

2c) $\int \frac{e^{2x}}{4+e^{4x}}\,dx$

$a=2$

$u=e^{2x}$

$\frac{du}{dx}=2e^{2x}$

$\frac{1}{2e^{2x}}du=dx$

$$\int \frac{\cancel{e^{2x}}}{a^2+u^2}\,\frac{1}{2\cancel{e^{2x}}}\,du$$

$$\frac{1}{2}\int \frac{1}{a^2+u^2}\,du$$

$$\frac{1}{2}\cdot\frac{1}{a}\text{Arctan}\left(\frac{u}{a}\right)+C$$

$$\boxed{\frac{1}{4}\text{Arctan}\left(\frac{e^{2x}}{2}\right)+C}$$

2d) $\int_{\pi/2}^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx$

$a=1$

$u=\cos x$

$\frac{du}{dx}=-\sin x$

$\frac{1}{-\sin x}du=dx$

$$= \int_0^{-1} \frac{\cancel{\sin x}}{a^2+u^2}\cdot\frac{1}{\cancel{-\sin x}}\,du$$

$$= \int_0^{-1} \frac{1}{a^2+u^2}\,\frac{du}{-1}$$

$$= -\frac{1}{a}\text{Arctan}\left(\frac{u}{a}\right)\Big|_0^{-1}$$

$$-\text{Arctan}(u)\Big|_0^{-1}$$

$$-(\text{Arctan}(-1)-\text{Arctan}(0))$$

$$-\left(-\frac{\pi}{4}-0\right)$$

$$\boxed{\frac{\pi}{4}}$$

2e) $\int \frac{x^3}{x^2+1}\,dx$

$$\begin{array}{r} x \\ x^2+1\,\overline{)\,x^3+0x^2+0x+0} \\ -(x^3 \qquad +x) \\ \hline -x \end{array}$$

$$= \int x - \frac{x}{x^2+1}\,dx$$

$$\int x\,dx - \int \frac{x}{x^2+1}\,dx$$

$$\frac{x^2}{2} -$$

↑ NOT AN ARCTAN reversal (x in num)

$u=x^2+1$

$\frac{du}{dx}=2x$

$$\frac{x^2}{2} - \int \frac{x}{u}\cdot\frac{1}{2x}\,du$$

$$\frac{x^2}{2} - \frac{1}{2}\int \frac{1}{u}\,du$$

$$\frac{1}{2}x^2 - \frac{1}{2}\ln|u| + C$$

$$\boxed{\frac{1}{2}x^2 - \frac{1}{2}\ln(x^2+1)+C}$$

2f) $\int_0^{\ln 5} \frac{e^x}{1+e^{2x}}\,dx$

$a=1$

$u=e^x$

$\frac{du}{dx}=e^x$

$\frac{1}{e^x}du=dx$

$$= \int_1^5 \frac{\cancel{e^x}}{1+u^2}\cdot\frac{1}{\cancel{e^x}}\,du$$

$$= \int_1^5 \frac{1}{1+u^2}\,du$$

$$\frac{1}{1}\text{Arctan}\left(\frac{u}{1}\right)\Big|_1^5$$

$$\text{Arctan}(u)\Big|_1^5$$

$$\text{Arctan}(5)-\text{Arctan}(1)$$

$$\boxed{\text{Arctan}(5)-\frac{\pi}{4}}$$

or

$\approx .588$

3a) $\int_0^{1/\sqrt{2}} \frac{\arcsin x}{\sqrt{1-x^2}}\,dx$

Let $u = \arcsin x$, $du = \frac{1}{\sqrt{1-x^2}}\,dx$.

$$\int_0^{1/\sqrt{2}} \frac{\arcsin x}{\sqrt{1-x^2}}\,dx = \left[\frac{1}{2}\arcsin^2 x\right]_0^{1/\sqrt{2}} = \frac{\pi^2}{32} \approx 0.308$$

3b) $\int_0^{1/\sqrt{2}} \frac{\arccos x}{\sqrt{1-x^2}}\,dx$

$u = \arccos x$

$\frac{du}{dx} = \frac{-1}{\sqrt{1-x^2}}$

$-\sqrt{1-x^2}\,du = dx$

$$= \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \frac{u}{\cancel{\sqrt{1-x^2}}} \cdot -\cancel{\sqrt{1-x^2}}\,du$$

$$= -\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} u\,du$$

$$= -\left[\frac{u^2}{2}\Big|_{\frac{\pi}{2}}^{\frac{\pi}{4}}\right] = -\left[\frac{\left(\frac{\pi}{4}\right)^2}{2} - \frac{\left(\frac{\pi}{2}\right)^2}{2}\right]$$

$$-\left[\frac{\pi^2}{32} - \frac{\pi^2}{8}\right] = \boxed{\frac{3\pi^2}{32}}$$

4a) $\int_0^2 \frac{dx}{x^2 - 2x + 2}$

$$= \int_0^2 \frac{1}{1 + (x-1)^2}\,dx$$

$$= \left[\arctan(x-1)\right]_0^2 = \frac{\pi}{2}$$

$-x^2 - 4x$

$= -(x^2 + 4x)$

$= -(x^2 + 4x + 4 - 4)$

$= -((x+2)^2 - 4)$

$= 4 - (x+2)^2$

4b) $\int \frac{1}{\sqrt{-x^2 - 4x}}\,dx$

$$= \int \frac{1}{\sqrt{4-(x+2)^2}}\,dx$$

$a = 2$

$u = x+2$

$\frac{du}{dx} = 1$

$du = dx$

$$= \int \frac{1}{\sqrt{a^2 - u^2}}\,du$$

$$= \text{Arcsin}\left(\frac{u}{a}\right) + c$$

$$= \boxed{\text{Arcsin}\left(\frac{x+2}{2}\right) + c}$$

4c) $\int \frac{x}{x^4 + 2x^2 + 2}\,dx$

$$= \int \frac{x}{x^4 + 2x^2 + 1 - 1 + 2}\,dx$$

$$= \int \frac{x}{(x^2+1)^2 + 1}\,dx$$

$a = 1$

$u = x^2 + 1$

$\frac{du}{dx} = 2x$

$\frac{1}{2x}\,du = dx$

$$= \int \frac{\cancel{x}}{u^2 + a^2} \cdot \frac{1}{2\cancel{x}}\,du$$

$$= \frac{1}{2}\int \frac{1}{u^2 + a^2}\,du$$

$$= \frac{1}{2} \cdot \frac{1}{a}\text{Arctan}\left(\frac{u}{a}\right) + c$$

$a = 1 \rightarrow$

so... $= \boxed{\frac{1}{2}\text{Arctan}(x^2+1) + c}$

Solve the differential equation with the given solution.

5a) $\frac{dy}{dx} = \frac{3}{1 + x^2}$, $(0, 0)$

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$y = 3\int \frac{dx}{1 + x^2} = 3 \arctan x + C$

$(0, 0)$: $0 = 3 \arctan(0) + C \Rightarrow C = 0$

$y = 3 \arctan x$

5b) $\frac{dy}{dx} = \frac{1}{x\sqrt{x^2 - 4}}$, $(2, 1)$

$$1\,dy = \frac{1}{x\sqrt{x^2-4}}\,dx$$

$$\int 1\,dy = \int \frac{1}{x\sqrt{x^2-4}}\,dx$$

$a =$

$$y = \frac{1}{2}\text{Arcsec}\frac{|x|}{2} + c$$

$$1 = \frac{1}{2}\text{Arcsec}\frac{|2|}{2} + c$$

$$1 = \frac{1}{2}\text{Arcsec}(1) + c$$

$$1 = \frac{1}{2} \cdot 0 + c$$

$$1 = c$$

$$\boxed{y = \frac{1}{2}\text{Arcsec}\frac{|x|}{2} + 1}$$

6) AP MULTIPLE CHOICE EXAMPLES

1) $\int \frac{5}{1+x^2}dx = 5\int \frac{1}{1+x^2}\,dx \quad = 5\,\text{Arctan}\,x + C$

(A) $\frac{-10x}{\left(1+x^2\right)^2}+C$

(B) $\frac{5}{2x}\ln\left(1+x^2\right)+C$

(C) $5x-\frac{5}{x}+C$

[✓ (D) $5\arctan x+C$]

(E) $5\ln\left(1+x^2\right)+C$

2) Which of the following is equal to $\int \frac{1}{\sqrt{25-x^2}}dx$? $= \text{Arcsin}\,\frac{x}{5}+C$

$a=5$

[✓ (A) $\arcsin\frac{x}{5}+C$]

(B) $\arcsin x+C$

(C) $\frac{1}{5}\arcsin\frac{x}{5}+C$

(D) $\sqrt{25-x^2}+C$

(E) $2\sqrt{25-x^2}+C$

3) An antiderivative for $\frac{1}{x^2-2x+2}$ is

(A) $-(x^2-2x+2)^{-2}$

(B) $\ln(x^2-2x+2)$

(C) $\ln\left|\frac{x-2}{x+1}\right|$

(D) $\text{arcsec}(x-1)$

[✓ (E) $\arctan(x-1)$]

$= \int \frac{1}{x^2-2x+1-1+2}\,dx$

$= \int \frac{1}{(x-1)^2+1}\,dx$

$u = x-1$

$a = 1$

$\frac{du}{dx} = 1$

$du = dx$

$= \int \frac{1}{u^2+a^2}\,du$

$= \frac{1}{1}\,\text{Arctan}\left(\frac{u}{1}\right)+C$

$= \text{Arctan}\,(x-1)+C$

4) $\int_0^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}} =$

$a=2$

[✓ (A) $\frac{\pi}{3}$] (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{6}$ (D) $\frac{1}{2}\ln 2$ (E) $-\ln 2$

$$= \text{Arcsin}\left(\frac{x}{2}\right)\Big|_0^{\sqrt{3}} = \text{Arcsin}\left(\frac{\sqrt{3}}{2}\right) - \text{Arcsin}(0)$$

$$\frac{\pi}{3} - 0$$

5) $\int_2^3 \frac{1}{x^2-4x+5}\,dx = \int_2^3 \frac{1}{x^2-4x+4-4+5} = \int_2^3 \frac{1}{(x-2)^2+1}\,dx$

[✓ (A) $\frac{\pi}{4}$] (B) $1-\frac{\pi}{4}$ (C) $1+\frac{\pi}{6}$ (D) $1+\frac{\pi}{4}$

$u = x-2$

$\frac{du}{dx} = 1$

$a = 1$

$$\int_0^1 \frac{1}{u^2+a^2}\,du$$

$$\frac{1}{1}\text{Arctan}\left(\frac{u}{1}\right)\Big|_0^1$$

$$\text{Arctan}(1) - \text{Arctan}(0)$$

$$\frac{\pi}{4} - 0$$

6) $\int_0^1 \frac{x^2}{x^2+1}\,dx =$

[✓ (A) $\frac{4-\pi}{4}$] (B) $\ln 2$ (C) 0 (D) $\frac{1}{2}\ln 2$ (E) $\frac{4+\pi}{4}$

$$\begin{array}{r} 1 \\ x^2+1\,\overline{)\,x^2+0x+0} \\ -(x^2 \qquad +1) \\ \hline 0+0x-1 \end{array}$$

So... $\int_0^1 1 - \frac{1}{x^2+1}\,dx$

$$\int_0^1 1\,dx - \int_0^1 \frac{1}{x^2+1}\,dx \quad a=1$$

$$x\Big|_0^1 - \frac{1}{1}\text{Arctan}\,x\Big|_0^1$$

$$(1-0) - (\text{Arctan}\,1 - \text{Arctan}(0))$$

$$1 - \left(\frac{\pi}{4} - 0\right)$$

$$\boxed{1 - \frac{\pi}{4}}$$

or

$\frac{4-\pi}{4}$ (subtract them)