Integrate each function.

1a) $\displaystyle\int \frac{x}{x^2-3}\,dx$

$u = x^2 - 3,\ du = 2x\,dx$

$$\int \frac{x}{x^2-3}\,dx = \frac{1}{2}\int \frac{1}{x^2-3}(2x)\,dx$$

$$= \frac{1}{2}\ln\left|x^2-3\right| + C$$

1b) $\displaystyle\int \frac{1}{2x+5}\,dx$

$u = 2x+5$

$\frac{du}{dx} = 2$

$\int \frac{1}{u}\cdot\frac{1}{2}du$

$\frac{1}{2}\int \frac{1}{u}\,du$

$\frac{1}{2}\ln|u| + C$

$\boxed{\frac{1}{2}\ln|2x+5| + C}$

1c) $\displaystyle\int \frac{4x^3+3}{x^4+3x}\,dx$

$u = x^4+3x$

$\frac{du}{dx} = 4x^3+3$

$\frac{1}{4x^3+3}du = dx$

$\int \frac{\cancel{4x^3+3}}{u}\cdot\frac{1}{\cancel{4x^3+3}}du$

$\int \frac{1}{u}\,du$

$\ln|u| + C$

$\boxed{\ln|x^4+3x| + C}$

1d) $\displaystyle\int \frac{\sec x\tan x}{\sec x - 1}\,dx$

$u = \sec x - 1$

$\frac{du}{dx} = \sec x\tan x$

$\frac{1}{\sec x\tan x}du = dx$

$\int \frac{\cancel{\sec x\tan x}}{u}\cdot\frac{1}{\cancel{\sec x\tan x}}du$

$\int \frac{1}{u}\,du$

$\ln|u| + C$

$\boxed{\ln|\sec x - 1| + C}$

1e) $\displaystyle\int_1^2 \frac{1-\cos\theta}{\theta-\sin\theta}\,d\theta$

$u = \theta - \sin\theta$

$\frac{du}{d\theta} = 1-\cos\theta$

$\frac{1}{1-\cos\theta}du = d\theta$

$\int_{1-\sin 1}^{2-\sin 2} \frac{\cancel{1-\cos\theta}}{u}\cdot\frac{1}{\cancel{1-\cos\theta}}du$

$\int_{1-\sin 1}^{2-\sin 2} \frac{1}{u}\,du$

$\ln|u|\Big|_{1-\sin 1}^{2-\sin 2}$

$\boxed{\ln|2-\sin 2| - \ln|1-\sin 1|} \approx -1.923$

2a) $\displaystyle\int \frac{x^2-3x+2}{x+1}\,dx$

USE LONG DIVISION!

$$= \int\left(x - 4 + \frac{6}{x+1}\right)dx$$

$$= \frac{x^2}{2} - 4x + 6\ln|x+1| + C$$

2b) $\displaystyle\int \frac{x^3-3x^2+5}{x-3}\,dx$

$$\begin{array}{r l}
 & x^2 \\
x-3 \,\big) & x^3-3x^2+0x+5 \\
 & -(x^3-3x^2) \\
\hline
 & 0 \quad +0x+5
\end{array}$$

$= \int x^2 + \frac{5}{x-3}\,dx$

$= \frac{x^3}{3} + \int \frac{5}{x-3}\,dx$ $\quad u = x-3$ $\quad \frac{du}{dx} = 1$

$= \frac{1}{3}x^3 + 5\int \frac{1}{u}\,du$

$= \frac{1}{3}x^3 + 5\cdot\ln|u| + C$

$= \boxed{\frac{1}{3}x^3 + 5\ln|x-3| + C}$

2c) $\displaystyle\int \frac{x^4+x-4}{x^2+2}\,dx$

$$\begin{array}{r l}
 & x^2-2 \\
x^2+2 \,\big) & x^4+0x^3+0x^2+x-4 \\
 & -(x^4 \quad +2x^2) \\
\hline
 & -2x^2+x-4 \\
 & (-2x^2 \quad -4) \\
\hline
 & x
\end{array}$$

$= \int x^2 - 2 + \frac{x}{x^2+2}$

$= \frac{x^3}{3} - 2x + \int \frac{x}{x^2+2}$ $\quad u = x^2+2$ $\quad \frac{du}{dx} = 2x$

$= \frac{1}{3}x^3 - 2x + \int \frac{\cancel{x}}{u}\frac{1}{2\cancel{x}}du$ $\quad \frac{1}{2x}du = dx$

$= \frac{1}{3}x^3 - 2x + \frac{1}{2}\int \frac{1}{u}\,du$

$= \frac{1}{3}x^3 - 2x + \frac{1}{2}\ln|u| + C$

$= \boxed{\frac{1}{3}x^3 - 2x + \frac{1}{2}\ln(x^2+2) + C}$

always positive

2d) $\int_0^2 \frac{x^2-2}{x+1}\,dx$

$$\begin{array}{r} x-1 \\ x+1 \overline{) x^2+0x-2} \\ -(x^2+1x) \\ \hline -x-2 \\ -(-x-1) \\ \hline -1 \end{array}$$

$= \int_0^2 x-1-\frac{1}{x+1}\,dx$

$= \int_0^2 x-1\,dx - \int_0^2 \frac{1}{x+1}\,dx$

$\frac{x^2}{2}-x\Big|_0^2 - \int_0^2 \frac{1}{x+1}\,dx$

$(2-2)-(0-0)$

$0 - \int_0^2 \frac{1}{x+1}\,dx$

$0 - \int_1^3 \frac{1}{u}\,du$ $\quad u=x+1$ $\quad \frac{du}{dx}=1$

$0 - \left(\ln|u|\Big|_1^3\right)$

$0 - (\ln 3 - \ln 1)$ zero

$\boxed{-\ln 3}$

DON'T BE ROBOTIC

2e) $\int \frac{x^2-4}{x}\,dx$ $= \int \frac{x^2}{x} - \frac{4}{x}\,dx$

$= \int x - \frac{4}{x}\,dx$

$= \frac{x^2}{2} - 4\int \frac{1}{x}\,dx$

$= \boxed{\frac{1}{2}x^2 - 4\ln|x| + C}$

3a) $\int \frac{1}{x \ln x^3}\,dx$

$= \int \frac{1}{x \bullet 3\ln x}\,dx = \frac{1}{3}\int \frac{1}{x\ln x}\,dx$

Let $u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

So... $\frac{1}{3}\int \frac{1}{x \bullet u} \bullet x\,du = \frac{1}{3}\int \frac{1}{u}\,du$

$= \frac{1}{3}\ln|u| + C$

$= \frac{1}{3}\ln|\ln x| + C$

3b) $\int \frac{(\ln x)^2}{x}\,dx$ $\quad u=\ln x$ $\quad \frac{du}{dx}=\frac{1}{x}$ $\quad x\,du = dx$

$\int \frac{u^2}{\cancel{x}} \cdot \cancel{x}\,du$

$\int u^2\,du$

$\frac{1}{3}u^3 + C$

$\boxed{\frac{1}{3}(\ln x)^3 + C}$

3c) $\int_1^e \frac{(1+\ln x)^2}{x}\,dx$ $\quad u = 1+\ln x$ $\quad \frac{du}{dx} = \frac{1}{x}$ $\quad x\,du = dx$

$\int_1^2 \frac{u^2}{\cancel{x}} \cancel{x}\,du$

$\int_1^2 u^2\,du$

$\frac{u^3}{3}\Big|_1^2$

$\frac{8}{3} - \frac{1}{3} = \boxed{\frac{7}{3}}$

4a) $\int \csc 2x\,dx$

$\int \csc(2x)\,dx$

Let $u = 2x$

$\frac{du}{dx} = 2$

So... $\frac{1}{2}\csc(u)\,du$

$= -\frac{1}{2}\ln|\csc(u)+\cot(u)| + C$

$= -\frac{1}{2}\ln|\csc(2x)+\cot(2x)| + C$

BOTH ARE "NEW" ONES WE KNOW!

4b) $\int (\sec(2x) + \tan(2x))\,dx$

$\int \sec(2x)\,dx + \int \tan(2x)\,dx$

in both integrals $\rightarrow$ $u = 2x$ $\quad \frac{du}{dx} = 2$ $\quad \frac{1}{2}du = dx$

$= \frac{1}{2}\int \sec(u)\,du + \frac{1}{2}\int \tan(u)\,du$

$= \frac{1}{2}\ln|\sec u + \tan u| - \frac{1}{2}\ln|\cos u|\,du$

$\boxed{= \frac{1}{2}\ln|\sec(2x)+\tan(2x)| - \frac{1}{2}\ln|\cos(2x)| + c}$

NEW ONE WE KNOW

4c) $\int \cot\left(\frac{\theta}{3}\right)d\theta$ $\quad u = \frac{1}{3}\theta$ $\quad \frac{du}{d\theta} = \frac{1}{3}$ $\quad 3\,du = d\theta$

$3\int \cot u\,du$

$= 3\ln|\sin u| + c$

$\boxed{= 3\ln|\sin\frac{\theta}{3}| + c}$

Solve the differential equation with the given solution.

5a) $\frac{dy}{dx} = \frac{3}{2-x}$, $(1, 0)$

$y = \int \frac{3}{2-x}\,dx$

$= -3\int \frac{1}{x-2}\,dx$

$= -3\ln|x-2| + C$

$(1, 0)$: $0 = -3\ln|1-2| + C \Rightarrow C = 0$

$y = -3\ln|x-2|$

5b) $\frac{dy}{dx} = \frac{1}{x+2}$, $(0, 1)$

$dy = \frac{1}{x+2}\,dx$

$\int dy = \int \frac{1}{x+2}\,dx$

$u = x+2$

$du = dx$

$y = \int \frac{1}{u}\,du$

$y = \ln|u| + c$

$y = \ln|x+2| + c$

$1 = \ln|0+2| + c$

$\rightarrow 1 - \ln 2 = c$

$y = \ln|x+2| + 1 - \ln 2$

or

$y = \ln\left(\frac{x+2}{2}\right) + 1$

5c) $\frac{ds}{d\theta} = \tan 2\theta$, $(0, 2)$

$ds = \tan(2\theta)\,d\theta$

$\int ds = \int \tan(2\theta)\,d\theta$

$u = 2\theta$

$\frac{du}{d\theta} = 2$

$s = \int \tan u \cdot \frac{1}{2}\,du$

$s = -\frac{1}{2}\ln|\cos u| + c$

$= -\frac{1}{2}\ln|\cos(2\theta)| + c$

$2 = -\frac{1}{2} \cdot \ln 1 + c$

$2 = 0 + c \rightarrow \boxed{s = -\frac{1}{2}\ln|\cos 2\theta| + 2}$

5d) $\frac{dy}{dx} = 1 + \frac{1}{x}$, $(1, 4)$

$dy = \left(1 + \frac{1}{x}\right)dx$

$\int dy = \int \left(1 + \frac{1}{x}\right)dx$

$y = x + \ln|x| + C$

$4 = 1 + \ln|1| + C$

$4 = 1 + 0 + C$

$3 = C$

$\boxed{y = x + \ln|x| + 3}$

6) AP MULTIPLE CHOICE EXAMPLES

1) Let $F(x)$ be an antiderivative of $\frac{(\ln x)^3}{x}$. If $F(1) = 0$, then $F(9) =$

(A) 0.048 (B) 0.144 [✓ (C) 5.827] (D) 23.308 (E) 1,640.250

$F(x) = \int \frac{(\ln x)^3}{x}\,dx$

$u = \ln x$

$F(x) = \int \frac{u^3}{\cancel{x}} \cdot \cancel{x} \cdot du$

$\frac{du}{dx} = \frac{1}{x}$

$du = \frac{1}{x}\,dx$

$F(x) = \int u^3\,du$

$x \cdot du = dx$

$F(x) = \frac{1}{4}u^4 + C$

$\rightarrow F(x) = \frac{1}{4}(\ln x)^4 + C$

$0 = \frac{1}{4}(\ln(1))^4 + C$

$0 = \frac{1}{4} \cdot 0 + C$

$0 = C$

$F(x) = \frac{1}{4}(\ln x)^4$

$F(9) = \frac{1}{4}(\ln 9)^4$ ✓ use calculator

$\approx \boxed{5.827}$

2) If $\frac{dy}{dx} = \tan x$, then $y =$

(A) $\frac{1}{2}\tan^2 x + C$ (B) $\sec^2 x + C$ [✓ (C) $\ln|\sec x| + C$]

(D) $\ln|\cos x| + C$ (E) $\sec x \tan x + C$

$-\ln|\cos x| = \ln|\cos x|^{-1}$
$= \ln|\frac{1}{\cos x}|$

$\frac{dy}{dx} = \tan x$

$dy = \tan x\,dx$

$\int dy = \int \tan x\,dx$ ←

$\boxed{y = -\ln|\cos x| + c}$

one of the new ones we need to know!

NOTE: DOES NOT MATCH "D"

3) $\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin x}\,dx =$

[✓ (A) $\ln\sqrt{2}$]   (B) $\ln\frac{\pi}{4}$ (C) $\ln\sqrt{3}$ (D) $\ln\frac{\sqrt{3}}{2}$ (E) $\ln e$

$\int_{\pi/4}^{\pi/2} \cot x\,dx = \ln|\sin x|\Big|_{\pi/4}^{\pi/2} = \ln|\sin\frac{\pi}{2}| - \ln|\sin\frac{\pi}{4}|$

$\ln|1| - \ln|\frac{\sqrt{2}}{2}|$

$0 - \ln\frac{\sqrt{2}}{2}$

$= -\ln(\frac{\sqrt{2}}{2}) = \ln(\frac{\sqrt{2}}{2})^{-1}$

$= \ln(\frac{2}{\sqrt{2}}) = \ln\frac{2\sqrt{2}}{2}$

$= \boxed{\ln\sqrt{2}}$

BRUTAL TRICKY FINISH

one of the new ones we need to know!

4) $\int_1^2 \frac{x-4}{x^2}\,dx =$

(A) $-\frac{1}{2}$ [✓ (B) $\ln 2 - 2$]   (C) $\ln 2$ (D) 2 (E) $\ln 2 + 2$

$\int_1^2 \frac{x}{x^2} - \frac{4}{x^2}\,dx$

$= \int_1^2 \frac{1}{x} - 4x^{-2}\,dx$

$\ln|x| + 4x^{-1}\Big|_1^2$

$(\ln 2 + 4(2)^{-1}) - (\ln 1 + 4(1)^{-1})$

$(\ln 2 + 2) - (0 + 4)$

$= \boxed{\ln 2 - 2}$

5) If $\frac{dy}{dx} = 4y$ and if $y = 4$ when $x = 0$, then $y =$

[✓ (A) $4e^{4x}$]   (B) e^{4x} (C) $3 + e^{4x}$ (D) $4 + e^{4x}$ (E) $2x^2 + 4$

$\frac{dy}{dx} = 4y$

$dy = 4y\,dx$

$\frac{1}{y}\,dy = 4\,dx$

$\int \frac{1}{y}\,dy = \int 4\,dx$

$\ln|y| = 4x + C$

$\ln|4| = 4(0) + C$

$\ln 4 = C$

$\ln|y| = 4x + \ln 4$

"solve for y"

$\log_e|y| = 4x + \ln 4$

$e^{4x+\ln 4} = |y|$

$\pm e^{4x+\ln 4} = y$

must be "+" because (0,4) on solution

"From Precalc" $e^{4x+\ln 4} = e^{4x} \cdot e^{\ln 4}$
$e^{4x} \cdot 4$

6) $\int_1^2 \frac{x+1}{x^2+2x}\,dx =$

(A) $\ln 8-\ln 3$ [✓ (B) $\frac{\ln 8-\ln 3}{2}$] (C) $\ln 8$ (D) $\frac{3\ln 2}{2}$ (E) $\frac{3\ln 2+2}{2}$

$\int_3^8 \frac{x+1}{u}\,dx$ $\quad u = x^2+2x$

$\frac{du}{dx} = 2x+2$

$\int_3^8 \frac{\cancel{x+1}}{u}\cdot\frac{1}{2\cancel{(x+1)}}\,du$ $\quad \frac{1}{2(x+1)}\,du = dx$

$\int_3^8 \frac{1}{2}\cdot\frac{1}{u}\,du$

$\frac{1}{2}\int_3^8 \frac{1}{u}\,du$

$\frac{1}{2}\left(\ln|u|\Big|_3^8\right)$

$\frac{1}{2}(\ln 8-\ln 3)$

7) At each point (x, y) on a certain curve, the slope of the curve is, $3x^2y$. If the curve contains the point $(0,8)$, then its equation is

$\frac{dy}{dx}$

[✓ (A) $y = 8e^{x^3}$] (B) $y = x^3+8$ (C) $y = e^{x^3}+7$

(D) $y = \ln(x+1)+8$ (E) $y^2 = x^3+8$

$\frac{dy}{dx} = 3x^2y$

$\frac{1}{y}\,dy = 3x^2\,dx$

$\int \frac{1}{y}\,dy = \int 3x^2\,dx$

$\ln|y| = \cancel{3}\cdot\frac{x^3}{\cancel{3}}+C$

$\ln|8| = 0^3+C$

$\ln 8 = C$

$\ln|y| = x^3+\ln 8$

$\log_e|y| = x^3+\ln 8$

$e^{x^3+\ln 8} = |y|$

$\pm e^{x^3+\ln 8} = y$

must go through (0,8)

$y = e^{x^3+\ln 8}$

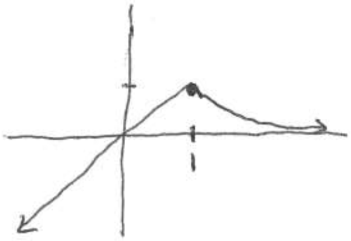
$y = e^{x^3}\cdot e^{\ln 8}$

$y = e^{x^3}\cdot 8$

$y = 8e^{x^3}$

8) If $f(x) = \begin{cases} x & \text{for } x \le 1 \\ \frac{1}{x} & \text{for } x > 1, \end{cases}$ then $\int_0^e f(x)\,dx =$

(A) 0 [✓ (B) $\frac{3}{2}$] (C) 2 (D) e (E) $e+\frac{1}{2}$



$\int_0^e f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^e f(x)\,dx$

$= \int_0^1 x\,dx + \int_1^e \frac{1}{x}\,dx$

$= \frac{x^2}{2}\Big|_0^1 + \ln|x|\Big|_1^e$

$\left(\frac{1}{2}-0\right) + (\ln e - \ln 1)$

$\frac{1}{2} + (1-0)$

$\frac{1}{2}+1 = \boxed{3/2}$