

Section 3.3 Increasing and Decreasing Functions and the First Derivative Test

1. $f(x) = x^2 - 6x + 8$

Increasing on: $(3, \infty)$ Decreasing on: $(-\infty, 3)$

3. $y = \frac{x^3}{4} - 3x$

Increasing on: $(-\infty, -2), (2, \infty)$ Decreasing on: $(-2, 2)$

5. $f(x) = \frac{1}{x^2}$

Increasing on: $(-\infty, 0)$ Decreasing on: $(0, \infty)$

7. $f(x) = -2x^2 + 4x + 3$

$f'(x) = -4x + 4 = 0$

Critical number: $x = 1$

Test intervals:	$-\infty < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$
Conclusion:	Increasing	Decreasing

Increasing on: $(-\infty, 1)$ Decreasing on: $(1, \infty)$ Relative maximum: $(1, 5)$

9. $f(x) = x^2 - 6x$

$f'(x) = 2x - 6 = 0$

Critical number: $x = 3$

Test intervals:	$-\infty < x < 3$	$3 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Increasing on: $(3, \infty)$ Decreasing on: $(-\infty, 3)$ Relative minimum: $(3, -9)$

2. $y = -(x + 1)^2$

Increasing on: $(-\infty, -1)$ Decreasing on: $(-1, \infty)$

4. $f(x) = x^4 - 2x^2$

Increasing on: $(-1, 0), (1, \infty)$ Decreasing on: $(-\infty, -1), (0, 1)$

6. $y = \frac{x^2}{x + 1}$

Increasing on: $(-\infty, -2), (0, \infty)$ Decreasing on: $(-2, -1), (-1, 0)$

8. $f(x) = x^2 + 8x + 10$

$f'(x) = 2x + 8 = 0$

Critical number: $x = -4$

Test intervals:	$-\infty < x < -4$	$-4 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Increasing on: $(-4, \infty)$ Decreasing on: $(-\infty, -4)$ Relative minimum: $(-4, -6)$

10. $f(x) = (x - 1)^2(x + 2)$

$$f'(x) = (x - 1)^2(1) + (x + 2)(2)(x - 1) = (x - 1)[(x - 1) + 2(x + 2)] = 3(x - 1)(x + 1) = 0$$

Critical numbers: $x = -1, 1$

Test intervals:	$-\infty < x < -1$	$-1 < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $(-\infty, -1), (1, \infty)$ Decreasing on: $(-1, 1)$ Relative maximum: $(-1, 4)$ Relative minimum: $(1, 0)$

11. $f(x) = 2x^3 + 3x^2 - 12x$

$$f'(x) = 6x^2 + 6x - 12 = 6(x + 2)(x - 1) = 0$$

Critical numbers: $x = -2, 1$

Test intervals:	$-\infty < x < -2$	$-2 < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $(-\infty, -2), (1, \infty)$ Decreasing on: $(-2, 1)$ Relative maximum: $(-2, 20)$ Relative minimum: $(1, -7)$

12. $f(x) = (x - 3)^3$

$$f'(x) = 3(x - 3)^2 = 0$$

Critical number: $x = 3$

Test intervals:	$-\infty < x < 3$	$-3 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' > 0$
Conclusion:	Increasing	Increasing

Increasing on: $(-\infty, \infty)$

No relative extrema

13. $f(x) = \frac{x^5 - 5x}{5}$

$$f'(x) = x^4 - 1$$

Critical numbers: $x = -1, 1$

Test intervals:	$-\infty < x < -1$	$-1 < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $(-\infty, -1), (1, \infty)$

Decreasing on: $(-1, 1)$

Relative maximum: $(-1, \frac{4}{5})$

Relative minimum: $(1, -\frac{4}{5})$

14. $f(x) = x^4 - 32x + 4$

$$f'(x) = 4x^3 - 32 = 4(x^3 - 8)$$

Critical number: $x = 2$

Test intervals:	$-\infty < x < 2$	$2 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Increasing on: $(2, \infty)$

Decreasing on: $(-\infty, 2)$

Relative minimum: $(2, -44)$

15. $f(x) = x^{1/3} + 1$

$$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$$

Critical number: $x = 0$

Test intervals:	$-\infty < x < 0$	$0 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' > 0$
Conclusion:	Increasing	Increasing

Increasing on: $(-\infty, \infty)$

No relative extrema

16. $f(x) = x^{2/3}(x - 5) = x^{5/3} - 5x^{2/3}$

$$f'(x) = \frac{5}{3}x^{2/3} - \frac{10}{3}x^{-1/3} = \frac{5}{3}x^{-1/3}(x - 2)$$

Critical numbers: $x = 0, 2$

Test intervals:	$-\infty < x < 0$	$0 < x < 2$	$2 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $(-\infty, 0), (2, \infty)$

Decreasing on: $(0, 2)$

Relative maximum: $(0, 0)$

Relative minimum: $(2, -3\sqrt[3]{4})$

17. $f(x) = 5 - |x - 5|$

$$f'(x) = -\frac{x-5}{|x-5|} = \begin{cases} 1, & x < 5 \\ -1, & x > 5 \end{cases}$$

Critical number: $x = 5$

Test intervals:	$-\infty < x < 5$	$5 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$
Conclusion:	Increasing	Decreasing

Increasing on: $(-\infty, 5)$ Decreasing on: $(5, \infty)$ Relative maximum: $(5, 5)$

18. $f(x) = |x + 3| - 1$

$$f'(x) = \frac{x+3}{|x+3|} = \begin{cases} 1, & x > -3 \\ -1, & x < -3 \end{cases}$$

Critical number: $x = -3$

Test intervals:	$-\infty < x < -3$	$-3 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Increasing on: $(-3, \infty)$ Decreasing on: $(-\infty, -3)$ Relative minimum: $(-3, -1)$

19. $f(x) = \frac{x^2}{x^2 - 9}$

$$f'(x) = \frac{(x^2 - 9)(2x) - (x^2)(2x)}{(x^2 - 9)^2} = \frac{-18x}{(x^2 - 9)^2}$$

Critical number: $x = 0$ Discontinuities: $x = -3, 3$

Test intervals:	$-\infty < x < -3$	$-3 < x < 0$	$0 < x < 3$	$3 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' > 0$	$f' < 0$	$f' < 0$
Conclusion:	Increasing	Increasing	Decreasing	Decreasing

Increasing on: $(-\infty, -3), (-3, 0)$ Decreasing on: $(0, 3), (3, \infty)$ Relative maximum: $(0, 0)$

20. $f(x) = \frac{x+3}{x^2} = \frac{1}{x} + \frac{3}{x^2}$

$$f'(x) = -\frac{1}{x^2} - \frac{6}{x^3} = \frac{-(x+6)}{x^3}$$

Critical number: $x = -6$ Discontinuity: $x = 0$

Test intervals:	$-\infty < x < -6$	$-6 < x < 0$	$0 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$	$f' < 0$
Conclusion:	Decreasing	Increasing	Decreasing

Increasing on: $(-6, 0)$ Decreasing on: $(-\infty, -6), (0, \infty)$ Relative minimum: $\left(-6, -\frac{1}{12}\right)$

21. $f(x) = x^3 - 6x^2 + 15$

$$f'(x) = 3x^2 - 12x = 3x(x - 4) = 0$$

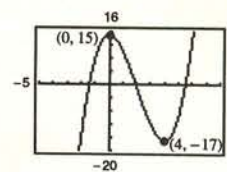
 Critical numbers: $x = 0, 4$

Test intervals:	$-\infty < x < 0$	$0 < x < 4$	$4 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

 Increasing on: $(-\infty, 0), (4, \infty)$

 Decreasing on: $(0, 4)$

 Relative maximum: $(0, 15)$

 Relative minimum: $(4, -17)$


22. $f(x) = x^4 - 2x^3$

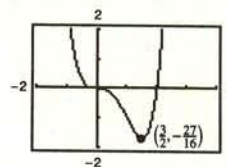
$$f'(x) = 4x^3 - 6x^2 = 2x^2(2x - 3) = 0$$

 Critical numbers: $x = 0, \frac{3}{2}$

Test intervals:	$-\infty < x < 0$	$0 < x < \frac{3}{2}$	$\frac{3}{2} < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Decreasing	Increasing

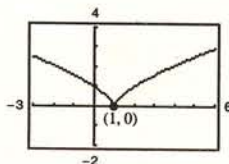
 Decreasing on: $(-\infty, \frac{3}{2})$

 Increasing on: $(\frac{3}{2}, \infty)$

 Relative minimum: $(\frac{3}{2}, -\frac{27}{16})$


23. $f(x) = (x - 1)^{2/3}$

$$f'(x) = \frac{2}{3(x - 1)^{1/3}}$$

 Critical number: $x = 1$


Test intervals:	$-\infty < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

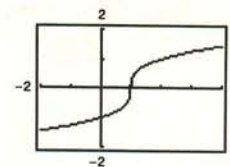
 Increasing on: $(1, \infty)$

 Decreasing on: $(-\infty, 1)$

 Relative minimum: $(1, 0)$

24. $f(x) = (x - 1)^{1/3}$

$$f'(x) = \frac{1}{3(x - 1)^{2/3}}$$

 Critical number:
 $x = 1$


Test intervals:	$-\infty < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' > 0$
Conclusion:	Increasing	Increasing

 Increasing on: $(-\infty, \infty)$

No relative extrema

25. $f(x) = x + \frac{1}{x}$

$$f'(x) = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2}$$

Critical numbers: $x = -1, 1$

Discontinuity: $x = 0$

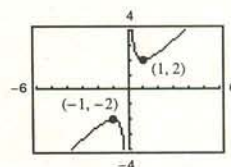
Test intervals:	$-\infty < x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Decreasing	Increasing

Increasing on: $(-\infty, -1), (1, \infty)$

Decreasing on: $(-1, 0), (0, 1)$

Relative maximum: $(-1, -2)$

Relative minimum: $(1, 2)$



26. $f(x) = \frac{x}{x+1}$

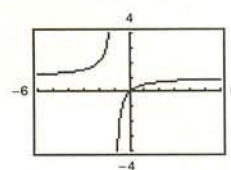
$$f'(x) = \frac{(x+1)(1) - (x)(1)}{(x+1)^2} = \frac{1}{(x+1)^2}$$

Discontinuity: $x = -1$

Test intervals:	$-\infty < x < -1$	$-1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' > 0$
Conclusion:	Increasing	Increasing

Increasing on: $(-\infty, -1), (-1, \infty)$

No relative extrema



27. $f(x) = \frac{x^2 - 2x + 1}{x+1}$

$$f'(x) = \frac{(x+1)(2x-2) - (x^2-2x+1)(1)}{(x+1)^2} = \frac{x^2+2x-3}{(x+1)^2} = \frac{(x+3)(x-1)}{(x+1)^2}$$

Critical numbers: $x = -3, 1$

Discontinuity: $x = -1$

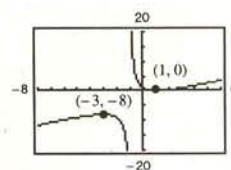
Test intervals:	$-\infty < x < -3$	$-3 < x < -1$	$-1 < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Decreasing	Increasing

Increasing on: $(-\infty, -3), (1, \infty)$

Decreasing on: $(-3, -1), (-1, 1)$

Relative maximum: $(-3, -8)$

Relative minimum: $(1, 0)$



28. $f(x) = \frac{x^2 - 3x - 4}{x - 2}$

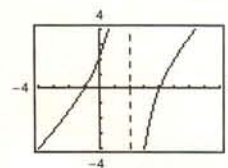
$$f'(x) = \frac{(x-2)(2x-3) - (x^2-3x-4)(1)}{(x-2)^2} = \frac{x^2 - 4x + 10}{(x-2)^2}$$

 Discontinuity: $x = 2$

Test intervals:	$-\infty < x < 2$	$2 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' > 0$
Conclusion:	Increasing	Increasing

 Increasing on: $(-\infty, 2), (2, \infty)$

No relative extrema



29. $f(x) = \frac{x}{2} + \cos x, 0 < x < 2\pi$

$$f'(x) = \frac{1}{2} - \sin x = 0$$

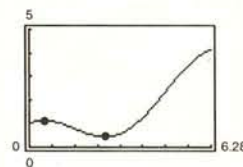
 Critical numbers: $x = \frac{\pi}{6}, \frac{5\pi}{6}$

Test intervals:	$0 < x < \frac{\pi}{6}$	$\frac{\pi}{6} < x < \frac{5\pi}{6}$	$\frac{5\pi}{6} < x < 2\pi$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

 Increasing on: $(0, \frac{\pi}{6}), (\frac{5\pi}{6}, 2\pi)$

 Decreasing on: $(\frac{\pi}{6}, \frac{5\pi}{6})$

 Relative maximum: $(\frac{\pi}{6}, \frac{\pi + 6\sqrt{3}}{12})$

 Relative minimum: $(\frac{5\pi}{6}, \frac{5\pi - 6\sqrt{3}}{12})$


30. $f(x) = \sin x \cos x = \frac{1}{2} \sin 2x, 0 < x < 2\pi$

$$f'(x) = \cos 2x = 0$$

Critical numbers: $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

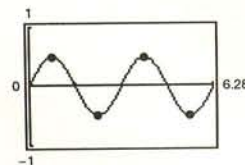
Test intervals:	$0 < x < \frac{\pi}{4}$	$\frac{\pi}{4} < x < \frac{3\pi}{4}$	$\frac{3\pi}{4} < x < \frac{5\pi}{4}$	$\frac{5\pi}{4} < x < \frac{7\pi}{4}$	$\frac{7\pi}{4} < x < 2\pi$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing	Decreasing	Increasing

Increasing on: $\left(0, \frac{\pi}{4}\right), \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right), \left(\frac{7\pi}{4}, 2\pi\right)$

Decreasing on: $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right), \left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)$

Relative maxima: $\left(\frac{\pi}{4}, \frac{1}{2}\right), \left(\frac{5\pi}{4}, \frac{1}{2}\right)$

Relative minima: $\left(\frac{3\pi}{4}, -\frac{1}{2}\right), \left(\frac{7\pi}{4}, -\frac{1}{2}\right)$



31. $f(x) = \sin^2 x + \sin x, 0 < x < 2\pi$

$$f'(x) = 2 \sin x \cos x + \cos x = \cos x(2 \sin x + 1) = 0$$

Critical numbers: $x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$

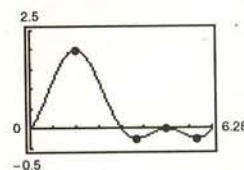
Test intervals:	$0 < x < \frac{\pi}{2}$	$\frac{\pi}{2} < x < \frac{7\pi}{6}$	$\frac{7\pi}{6} < x < \frac{3\pi}{2}$	$\frac{3\pi}{2} < x < \frac{11\pi}{6}$	$\frac{11\pi}{6} < x < 2\pi$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing	Decreasing	Increasing

Increasing on: $\left(0, \frac{\pi}{2}\right), \left(\frac{7\pi}{6}, \frac{3\pi}{2}\right), \left(\frac{11\pi}{6}, 2\pi\right)$

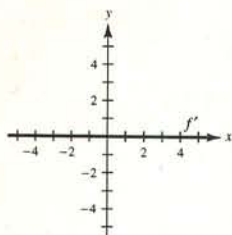
Decreasing on: $\left(\frac{\pi}{2}, \frac{7\pi}{6}\right), \left(\frac{3\pi}{2}, \frac{11\pi}{6}\right)$

Relative minima: $\left(\frac{7\pi}{6}, -\frac{1}{4}\right), \left(\frac{11\pi}{6}, -\frac{1}{4}\right)$

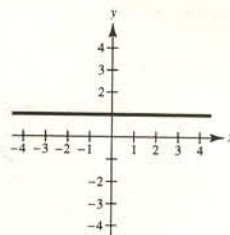
Relative maxima: $\left(\frac{\pi}{2}, 2\right), \left(\frac{3\pi}{2}, 0\right)$



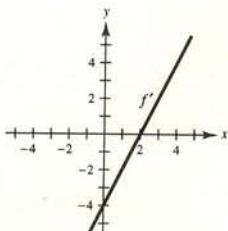
43. $f(x) = c$ is constant $\Rightarrow f'(x) = 0$



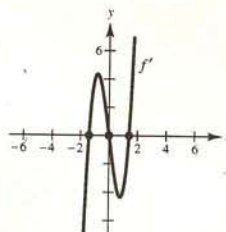
44. $f(x)$ is a line of slope $\approx 1 \Rightarrow f'(x) = 1$.



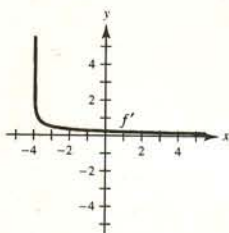
45. f is quadratic $\Rightarrow f'$ is a line.



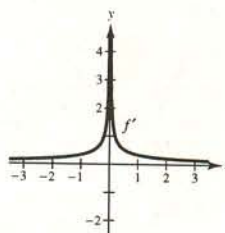
46. f is a 4th degree polynomial $\Rightarrow f'$ is a cubic polynomial.



47. f has positive, but decreasing slope



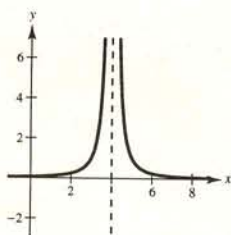
48. f has positive slope



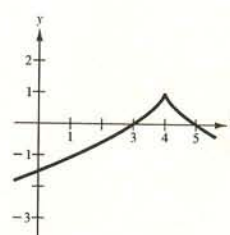
49. $f'(x) = \begin{cases} > 0, & x < 4 \Rightarrow f \text{ is increasing on } (-\infty, 4). \\ \text{undefined}, & x = 4 \\ < 0, & x > 4 \Rightarrow f \text{ is decreasing on } (4, \infty). \end{cases}$

 Two possibilities for $f(x)$ are given below.

(a)



(b)

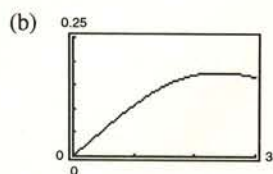


54. $C = \frac{3t}{27 + t^3}, t \geq 0$

(a)

t	0	0.5	1	1.5	2	2.5	3
$C(t)$	0	0.055	0.107	0.148	0.171	0.176	0.167

The concentration seems greater near $t = 2.5$ hours.



The concentration is greatest when $t \approx 2.38$ hours.

(c) $C' = \frac{(27 + t^3)(3) - (3t)(3t^2)}{(27 + t^3)^2}$
 $= \frac{3(27 - 2t^3)}{(27 + t^3)^2}$

$C' = 0$ when $t = 3/\sqrt[3]{2} \approx 2.38$ hours.

By the First Derivative Test, this is a maximum.

55. $v = k(R - r)r^2 = k(Rr^2 - r^3)$

$$v' = k(2Rr - 3r^2)$$

$$= kr(2R - 3r) = 0$$

$$r = 0 \text{ or } \frac{2}{3}R$$

Maximum when $r = \frac{2}{3}R$.

56. $P = 2.44x - \frac{x^2}{20,000} - 5000, 0 \leq x \leq 35,000$

$$P' = 2.44 - \frac{x}{10,000} = 0$$

$$x = 24,400$$

Increasing when $0 < x < 24,400$ hamburgers.

Decreasing when $24,400 < x < 35,000$ hamburgers.

Test intervals:	$0 < x < 24,400$	$24,400 < x < 35,000$
Sign of P' :	$P' > 0$	$P' < 0$

57. $P = \frac{vR_1R_2}{(R_1 + R_2)^2}, v \text{ and } R_1 \text{ are constant}$

$$\frac{dP}{dR_2} = \frac{(R_1 + R_2)^2(vR_1) - vR_1R_2[2(R_1 + R_2)(1)]}{(R_1 + R_2)^4}$$

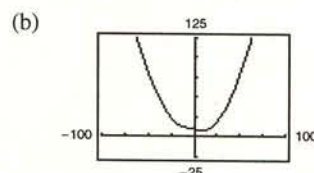
$$= \frac{vR_1(R_1 - R_2)}{(R_1 + R_2)^3} = 0 \Rightarrow R_2 = R_1$$

Maximum when $R_1 = R_2$.

58. $R = \sqrt{0.001T^4 - 4T + 100}$

(a) $R' = \frac{0.004T^3 - 4}{2\sqrt{0.001T^4 - 4T + 100}} = 0$

$$T = 10^\circ, R \approx 8.3666\Omega$$



The minimum resistance is approximately $R \approx 8.37\Omega$ at $T = 10^\circ$.