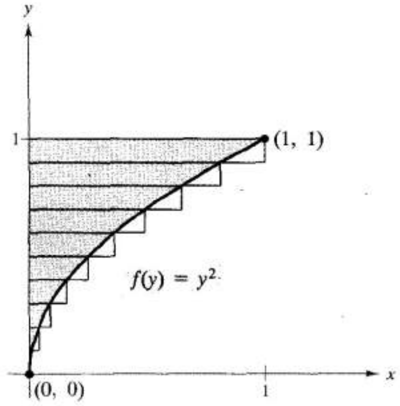
The last example in this section looks at a region that is bounded by the y-axis (rather than the x-axis).

EXAMPLE 7 A Region Bounded by the y-Axis

Find the area of the region bounded by the graph of $f(y) = y^2$ and the y-axis for $0 \le y \le 1$, as shown in Figure 4.17.



The area of the region bounded by the graph of f and the y-axis for $0 \le y \le 1$ is $\frac{1}{3}$.
Figure 4.17

Solution When f is a continuous, nonnegative function of y, you still can use the same basic procedure illustrated in Examples 5 and 6. Begin by partitioning the interval $[0, 1]$ into n equal subintervals, each of length $\Delta y = 1/n$. Then, using the upper endpoints $c_i = i/n$, you obtain the following.

$$\begin{aligned}
\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta y &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2\left(\frac{1}{n}\right) \\
&= \lim_{n\to\infty} \frac{1}{n^3}\sum_{i=1}^{n} i^2 \\
&= \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3} \\
&= \lim_{n\to\infty} \left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right) \\
&= \frac{1}{3}
\end{aligned}$$

Thus, the area of the region is $\frac{1}{3}$.

EXERCISES FOR SECTION 4.2

In Exercises 1–6, find the sum. Use the summation capabilities of a graphing utility to verify your result.

1. $\sum_{i=1}^{5} (2i + 1)$
2. $\sum_{k=2}^{5} (k + 1)(k - 3)$
3. $\sum_{k=0}^{4} \frac{1}{k^2 + 1}$
4. $\sum_{j=3}^{5} \frac{1}{j}$
5. $\sum_{k=1}^{4} c$
6. $\sum_{i=1}^{4} [(i - 1)^2 + (i + 1)^3]$

In Exercises 7–14, use sigma notation to write the sum.

7. $\frac{1}{3(1)} + \frac{1}{3(2)} + \frac{1}{3(3)} + \cdots + \frac{1}{3(9)}$
8. $\frac{5}{1+1} + \frac{5}{1+2} + \frac{5}{1+3} + \cdots + \frac{5}{1+15}$
9. $\left[2\left(\frac{1}{8}\right) + 3\right] + \left[2\left(\frac{2}{8}\right) + 3\right] + \cdots + \left[2\left(\frac{8}{8}\right) + 3\right]$
10. $\left[1 - \left(\frac{1}{4}\right)^2\right] + \left[1 - \left(\frac{2}{4}\right)^2\right] + \cdots + \left[1 - \left(\frac{4}{4}\right)^2\right]$
11. $\left[\left(\frac{2}{n}\right)^3 - \frac{2}{n}\right]\left(\frac{2}{n}\right) + \cdots + \left[\left(\frac{2n}{n}\right)^3 - \frac{2n}{n}\right]\left(\frac{2}{n}\right)$
12. $\left[1 - \left(\frac{2}{n} - 1\right)^2\right]\left(\frac{2}{n}\right) + \cdots + \left[1 - \left(\frac{2n}{n} - 1\right)^2\right]\left(\frac{2}{n}\right)$
13. $\left[2\left(1 + \frac{3}{n}\right)^2\right]\left(\frac{3}{n}\right) + \cdots + \left[2\left(1 + \frac{3n}{n}\right)^2\right]\left(\frac{3}{n}\right)$
14. $\left(\frac{1}{n}\right)\sqrt{1 + \left(\frac{0}{n}\right)^2} + \cdots + \left(\frac{1}{n}\right)\sqrt{1 - \left(\frac{n-1}{n}\right)^2}$

In Exercises 15–20, use the properties of sigma notation and summation formulas to evaluate the sum. Use the summation capabilities of a graphing utility to verify your result.

15. $\sum_{i=1}^{20} 2i$
16. $\sum_{i=1}^{15} (2i - 3)$

17. $\sum_{i=1}^{20} (i-1)^2$

18. $\sum_{i=1}^{10} (i^2-1)$

19. $\sum_{i=1}^{15} i(i-1)^2$

20. $\sum_{i=1}^{10} i(i^2+1)$

In Exercises 21 and 22, use the summation capabilities of a graphing utility to evaluate the sum. Then use the properties of sigma notation and summation formulas to verify the sum.

21. $\sum_{i=1}^{20} (i^2+3)$

22. $\sum_{i=1}^{15} (i^3-2i)$

In Exercises 23–28, find the limit of $s(n)$ as $n \to \infty$.

23. $s(n) = \left(\frac{4}{3n^3}\right)(2n^3 + 3n^2 + n)$

24. $s(n) = \left(\frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2}\right)$

25. $s(n) = \frac{81}{n^4}\left[\frac{n^2(n+1)^2}{4}\right]$

26. $s(n) = \frac{64}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$

27. $s(n) = \frac{18}{n^2}\left[\frac{n(n+1)}{2}\right]$

28. $s(n) = \frac{1}{n^2}\left[\frac{n(n+1)}{2}\right]$

In Exercises 29–34, find a formula for the sum of n terms. Use the formula to find the limit as $n \to \infty$.

29. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{16i}{n^2}$

30. $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2i}{n}\right)\left(\frac{2}{n}\right)$

31. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n^3}(i-1)^2$

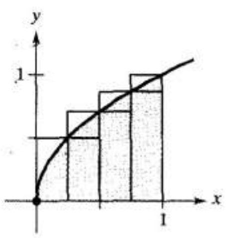
32. $\lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \frac{2i}{n}\right)^2\left(\frac{2}{n}\right)$

33. $\lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \frac{i}{n}\right)\left(\frac{2}{n}\right)$

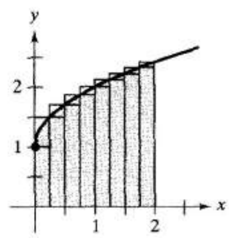
34. $\lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \frac{2i}{n}\right)^3\left(\frac{2}{n}\right)$

In Exercises 35–38, use upper and lower sums to approximate the area of the region using the indicated number of subintervals (of equal length).

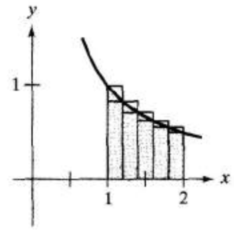
35. $y = \sqrt{x}$



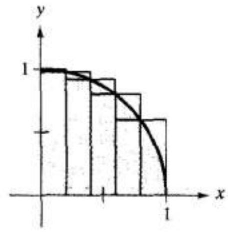
36. $y = \sqrt{x} + 1$



37. $y = \frac{1}{x}$



38. $y = \sqrt{1 - x^2}$



39. ***Numerical Reasoning*** Consider a triangle of area 2 bounded by the graphs of $y = x$, $y = 0$, and $x = 2$.

(a) Sketch the graph of the region.

(b) Divide the interval $[0, 2]$ into n equal subintervals and show that the endpoints are

$$0 < 1\left(\frac{2}{n}\right) < \cdots < (n-1)\left(\frac{2}{n}\right) < n\left(\frac{2}{n}\right).$$

(c) Show that $s(n) = \sum_{i=1}^{n}\left[(i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$.

(d) Show that $S(n) = \sum_{i=1}^{n}\left[i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$.

(e) Complete the table.

n	5	10	50	100
$s(n)$				
$S(n)$				

(f) Show that $\lim_{n\to\infty} s(n) = \lim_{n\to\infty} S(n) = 2$.

40. ***Numerical Reasoning*** Consider a trapezoid of area 4 bounded by the graphs of $y = x$, $y = 0$, $x = 1$, and $x = 3$.

(a) Sketch the graph of the region.

(b) Divide the interval $[1, 3]$ into n equal subintervals and show that the endpoints are

$$1 < 1 + 1\left(\frac{2}{n}\right) < \cdots < 1 + (n-1)\left(\frac{2}{n}\right) < 1 + n\left(\frac{2}{n}\right).$$

(c) Show that $s(n) = \sum_{i=1}^{n}\left[1 + (i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$.

(d) Show that $S(n) = \sum_{i=1}^{n}\left[1 + i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$.

(e) Complete the table.

n	5	10	50	100
$s(n)$				
$S(n)$				

(f) Show that $\lim_{n\to\infty} s(n) = \lim_{n\to\infty} S(n) = 4$.

In Exercises 41–48, use the limit process to find the area of the region between the graph of the function and the x-axis over the indicated interval. Sketch the region.

Function	*Interval*
41. $y = -2x + 3$	$[0, 1]$
42. $y = 3x - 4$	$[2, 5]$
43. $y = x^2 + 2$	$[0, 1]$
44. $y = 1 - x^2$	$[-1, 1]$
45. $y = 27 - x^3$	$[1, 3]$
46. $y = 2x - x^3$	$[0, 1]$
47. $y = x^2 - x^3$	$[-1, 1]$
48. $y = x^2 - x^3$	$[-1, 0]$

In Exercises 49 and 50, use the limit process to find the area of the region between the graph of the function and the y-axis over the indicated y-interval. Sketch the region.

49. $f(y) = 3y, 0 \le y \le 2$

50. $f(y) = y^2, 0 \le y \le 3$

In Exercises 51–54, use the *Midpoint Rule*

$$\text{Area} \approx \sum_{i=1}^{n} f\left(\frac{x_i + x_{i-1}}{2}\right)\Delta x$$

with $n = 4$ to approximate the area of the region bounded by the graph of the function and the x-axis over the indicated interval.

Function	*Interval*
51. $f(x) = x^2 + 3$	$[0, 2]$
52. $f(x) = x^2 + 4x$	$[0, 4]$
53. $f(x) = \tan x$	$\left[0, \frac{\pi}{4}\right]$
54. $f(x) = \sin x$	$\left[0, \frac{\pi}{2}\right]$

Write a program for a graphing utility to approximate areas by using the Midpoint Rule. Assume the function is positive over the indicated interval and the subintervals are of equal width. In Exercises 55–58, use the program to approximate the area between the function and the x-axis over the indicated interval, and complete the table.

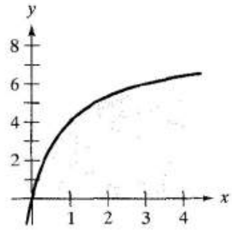
n	4	8	12	16	20
Approximate area					

Function	*Interval*
55. $f(x) = \sqrt{x}$	$[0, 4]$
56. $f(x) = \frac{8}{x^2 + 1}$	$[2, 6]$
57. $f(x) = \tan\left(\frac{\pi x}{8}\right)$	$[1, 3]$
58. $f(x) = \cos\sqrt{x}$	$[0, 2]$

59. ***Graphical Reasoning*** Consider the region bounded by the graphs of

$$f(x) = \frac{8x}{x + 1},$$

$x = 0, x = 4$, and $y = 0$, as shown in the figure.



(a) Redraw the figure, and complete and shade the rectangles representing the lower sum when $n = 4$. Find this lower sum.

(b) Redraw the figure, and complete and shade the rectangles representing the upper sum when $n = 4$. Find this upper sum.

(c) Redraw the figure, and complete and shade the rectangles whose heights are determined by the functional values at the midpoint of each subinterval when $n = 4$. Find this sum using the Midpoint Rule.

(d) Verify the following formulas for approximating the area of the region using n subintervals of equal width.

Lower sum: $s(n) = \sum_{i=1}^{n} f\left[(i - 1)\frac{4}{n}\right]\left(\frac{4}{n}\right)$

Upper sum: $S(n) = \sum_{i=1}^{n} f\left[(i)\frac{4}{n}\right]\left(\frac{4}{n}\right)$

Midpoint Rule: $M(n) = \sum_{i=1}^{n} f\left[\left(i - \frac{1}{2}\right)\frac{4}{n}\right]\left(\frac{4}{n}\right)$

(e) Use a graphing utility and the formulas in part (d) to complete the table.

n	4	8	20	100	200
$s(n)$					
$S(n)$					
$M(n)$					

(f) Explain why $s(n)$ increases and $S(n)$ decreases for increasing n, as shown in the table in part (e).

60. Use a graphing utility to complete the table for approximations of the area of the region bounded by the graphs of $f(x) = \sqrt[3]{x}$, $x = 0, x = 8$, and $y = 0$.

n	10	20	50	100	200
$s(n)$					
$S(n)$					
$M(n)$					

Approximation **In Exercises 61 and 62, determine which value best approximates the area of the region between the x-axis and the graph of the function over the indicated interval. (Make your selection on the basis of a sketch of the region and *not* by performing calculations.)**

61. $f(x) = 4 - x^2, \quad [0, 2]$

(a) -2 (b) 6 (c) 10 (d) 3 (e) 8

62. $f(x) = \sin \frac{\pi x}{4}, \quad [0, 4]$

(a) 3 (b) 1 (c) -2 (d) 8 (e) 6

True or False? **In Exercises 63 and 64, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

63. The sum of the first n positive integers is $n(n + 1)/2$.

64. If f is continuous and nonnegative on $[a, b]$, then the limits as $n \to \infty$ of its lower sum $s(n)$ and upper sum $S(n)$ both exist and are equal.

65. ***Monte Carlo Method*** The following program approximates the area of the region under the graph of a monotonic function and above the x-axis between $x = a$ and $x = b$. Run the program for $a = 0$ and $b = \pi/2$ for several values of N2. Explain why the Monte Carlo Method works. *(Adaptation of Monte Carlo Method program, from James M. Sconyers, "Approximation of Area Under a Curve," MATHEMATICS TEACHER 77, no. 2 (February 1984). Copyright © 1984 by the National Council of Teachers of Mathematics. Reprinted with permission.)*

```
10  DEF FNF(X)=SIN(X)
20  A=0
30  B=1.570796
40  PRINT "Input Number of Random Points"
50  INPUT N2
60  N1=0
70  IF FNF(A)>FNF(B) THEN YMAX=FNF(A) ELSE
    YMAX=FNF(B)
80  FOR I=1 TO N2
90  X=A+(B-A)*RND(1)
100 Y=YMAX*RND(1)
110 IF Y>=FNF(X) THEN GOTO 130
120 N1=N1+1
130 NEXT I
140 AREA=(N1/N2)*(B-A)*YMAX
150 PRINT "Approximate Area:"; AREA
160 END
```

66. ***Graphical Reasoning*** Consider an n-sided regular polygon inscribed in a circle of radius r. Join the vertices of the polygon to the center of the circle, forming n congruent triangles (see figure).

(a) Determine the central angle θ in terms of n.

(b) Show that the area of each triangle is $\frac{1}{2}r^2 \sin \theta$.

(c) Let A_n be the sum of the areas of the n triangles. Find $\lim_{n\to\infty} A_n$.

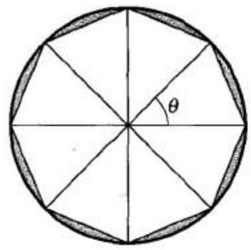


Figure for 66

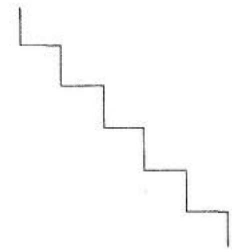


Figure for 67

67. ***Writing*** Use the figure to write a short paragraph explaining why the formula

$$1 + 2 + \cdots + n = \tfrac{1}{2}n(n + 1)$$

is valid for all positive integers n.

68. Verify the formula

$$\sum_{i=1}^{n} i^2 = \frac{n(n + 1)(2n + 1)}{6}$$

by showing the following.

(a) $(1 + i)^3 - i^3 = 3i^2 + 3i + 1$

(b) $-1 + (n + 1)^3 = \sum_{i=1}^{n} (3i^2 + 3i + 1)$

(c) $\sum_{i=1}^{n} i^2 = \dfrac{n(n + 1)(2n + 1)}{6}$

69. ***Modeling Data*** The table lists the measurements of a lot bounded by a stream and two straight roads that meet at right angles, where x and y are measured in feet (see figure).

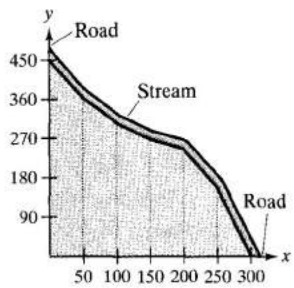
x	0	50	100	150	200	250	300
y	450	362	305	268	245	156	0

(a) Use the regression capabilities of a graphing utility to find a model of the form

$$y = ax^3 + bx^2 + cx + d.$$

(b) Use a graphing utility to plot the data and graph the model.

(c) Use the model in part (a) to estimate the area of the lot.



LAB SERIES Lab 4.1

EXERCISES FOR SECTION 4.6

In Exercises 1–10, use the Trapezoidal Rule and Simpson's Rule to approximate the value of the definite integral for the indicated value of n. Round your answer to four decimal places and compare the results with the exact value of the definite integral.

1. $\int_0^2 x^2\,dx,\quad n = 4$
2. $\int_0^1 \left(\frac{x^2}{2} + 1\right) dx,\quad n = 4$
3. $\int_0^2 x^3\,dx,\quad n = 4$
4. $\int_1^2 \frac{1}{x^2}\,dx,\quad n = 4$
5. $\int_0^2 x^3\,dx,\quad n = 8$
6. $\int_0^8 \sqrt[3]{x}\,dx,\quad n = 8$
7. $\int_4^9 \sqrt{x}\,dx,\quad n = 8$
8. $\int_1^3 (4 - x^2)\,dx,\quad n = 4$
9. $\int_1^2 \frac{1}{(x+1)^2}\,dx,\quad n = 4$
10. $\int_0^2 x\sqrt{x^2+1}\,dx,\quad n = 4$

In Exercises 11–20, approximate the definite integral using the Trapezoidal Rule and Simpson's Rule with $n = 4$. Compare these results with the approximation of the integral using a graphing utility.

11. $\int_0^2 \sqrt{1 + x^3}\,dx$
12. $\int_0^2 \frac{1}{\sqrt{1 + x^3}}\,dx$
13. $\int_0^1 \sqrt{x}\,\sqrt{1 - x}\,dx$
14. $\int_{\pi/2}^{\pi} \sqrt{x}\sin x\,dx$
15. $\int_0^{\sqrt{\pi/2}} \cos x^2\,dx$
16. $\int_0^{\sqrt{\pi/4}} \tan x^2\,dx$
17. $\int_1^{1.1} \sin x^2\,dx$
18. $\int_0^{\pi/2} \sqrt{1 + \cos^2 x}\,dx$
19. $\int_0^{\pi/4} x\tan x\,dx$
20. $\int_0^{\pi} f(x)\,dx,\quad f(x) = \begin{cases} \dfrac{\sin x}{x}, & x > 0 \\ 1, & x = 0 \end{cases}$

In Exercises 21 and 22, use the error formulas in Theorem 4.19 to find the maximum possible error in approximating the integral, with $n = 4$, using (a) the Trapezoidal Rule and (b) Simpson's Rule.

21. $\int_0^2 x^3\,dx$
22. $\int_0^1 \frac{1}{x+1}\,dx$

In Exercises 23 and 24, use the error formulas in Theorem 4.19 to find n such that the error in the approximation of the definite integral is less than 0.00001 using (a) the Trapezoidal Rule and (b) Simpson's Rule.

23. $\int_1^3 \frac{1}{x}\,dx$
24. $\int_0^1 \frac{1}{1+x}\,dx$

In Exercises 25–28, use a symbolic differentiation utility and the error formulas to find n such that the error in the approximation of the definite integral is less than 0.00001 using (a) the Trapezoidal Rule and (b) Simpson's Rule.

25. $\int_0^2 \sqrt{1 + x}\,dx$
26. $\int_0^2 (x+1)^{2/3}\,dx$
27. $\int_0^1 \tan x^2\,dx$
28. $\int_0^1 \sin x^2\,dx$

29. Prove that Simpson's Rule is exact when approximating the integral of a cubic polynomial function, and demonstrate the result for

$$\int_0^1 x^3\,dx,\quad n = 2.$$

30. Write a program for a graphing utility to approximate a definite integral using the Trapezoidal Rule and Simpson's Rule. Start with the program written in Section 4.3, Exercises 35–38, and note that the Trapezoidal Rule can be written as

$$T(n) = \tfrac{1}{2}\left[L(n) + R(n)\right]$$

and Simpson's Rule can be written as

$$S(n) = \tfrac{1}{3}\left[T(n/2) + 2M(n/2)\right].$$

[Recall that $L(n)$, $M(n)$, and $R(n)$ represent the Riemann sums using the left-hand endpoint, midpoint, and right-hand endpoint of subintervals of equal width.]

In Exercises 31–34, use the program in Exercise 30 to approximate the definite integral and complete the table.

n	$L(n)$	$M(n)$	$R(n)$	$T(n)$	$S(n)$
4					
8					
10					
12					
16					
20					

31. $\int_0^4 \sqrt{2 + 3x^2}\,dx$
32. $\int_0^1 \sqrt{1 - x^2}\,dx$
33. $\int_0^4 \sin\sqrt{x}\,dx$
34. $\int_1^2 \frac{\sin x}{x}\,dx$

35. ***Area*** Use Simpson's Rule with $n = 14$ to approximate the area of the region bounded by the graphs of $y = \sqrt{x}\cos x$, $y = 0$, $x = 0$, and $x = \pi/2$.

36. ***Circumference*** The **elliptic integral**

$$8\sqrt{3}\int_0^{\pi/2}\sqrt{1-\tfrac{2}{3}\sin^2\theta}\,d\theta$$

gives the circumference of an ellipse. Use Simpson's Rule with $n = 8$ to approximate the circumference.

37. ***Work*** To determine the size of the motor required to operate a press, a company must know the amount of work done when the press moves an object linearly 5 feet. The variable force to move the object is

$$F(x) = 100x\sqrt{125 - x^3}$$

where F is given in pounds and x gives the position of the unit in feet. Use Simpson's Rule with $n = 12$ to approximate the work W (in foot-pounds) done through one cycle if

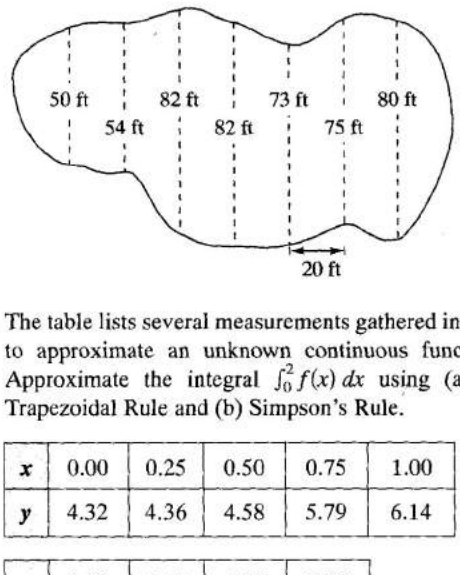
$$W = \int_0^5 F(x)\,dx.$$

38. ***Approximation of Pi*** Use Simpson's Rule with $n = 6$ to approximate π using the equation

$$\pi = \int_0^1 \frac{4}{1+x^2}\,dx.$$

(In Section 5.9, you will be able to evaluate this integral using the inverse tangent function.)

39. ***Area*** To estimate the surface area of a pond, a surveyor takes several measurements, as shown in the figure. Estimate the surface area of the pond using (a) the Trapezoidal Rule and (b) Simpson's Rule.



40. The table lists several measurements gathered in an experiment to approximate an unknown continuous function $y = f(x)$. Approximate the integral $\int_0^2 f(x)\,dx$ using (a) the Trapezoidal Rule and (b) Simpson's Rule.

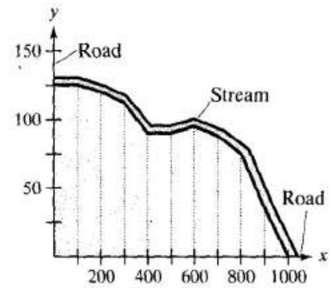
x	0.00	0.25	0.50	0.75	1.00
y	4.32	4.36	4.58	5.79	6.14

x	1.25	1.50	1.75	2.00
y	7.25	7.64	8.08	8.14

Area **In Exercises 41 and 42, use the Trapezoidal Rule to estimate the number of square meters of land in a lot where x and y are measured in meters, as shown in the figures. The land is bounded by a stream and two straight roads that meet at right angles.**

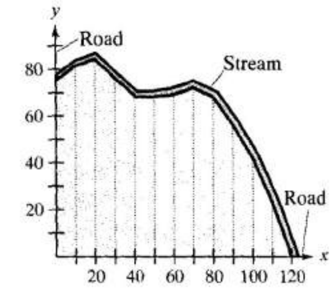
41.

x	y
0	125
100	125
200	120
300	112
400	90
500	90
600	95
700	88
800	75
900	35
1000	0



42.

x	y
0	75
10	81
20	84
30	76
40	67
50	68
60	69
70	72
80	68
90	56
100	42
110	23
120	0



43. Use Simpson's Rule with $n = 10$ and a symbolic integration utility to approximate t in the integral equation

$$\int_0^t \sin\sqrt{x}\,dx = 2.$$

44. Determine whether the Trapezoidal Rule overestimates or underestimates a definite integral if the graph of the integrand is (a) concave up and (b) concave down.

45. Let $L(x) = \int_1^x \frac{1}{t}\,dt$ for all $x > 0$.

(a) Find $L(1)$.

(b) Find $L'(x)$ and $L'(1)$.

(c) Use the Trapezoidal Rule to approximate the value of x (to three decimal places) for which $L(x) = 1$.

(d) Prove that $L(x_1x_2) = L(x_1) + L(x_2)$, for $x_1 > 0$ and $x_2 > 0$.